Anisotropic cosmology in Sáez-Ballester theory: classical and quantum solutions

J. Socorro[1],* M. Sabido[1],† M.A. Sánchez G.[2], and M.G. Frías Palos[2]
[1] *Departamento de Física, DCeI, Universidad de Guanajuato-Campus León A.P. E-143, C.P. 37150, Guanajuato, México*
[2] *Facultad de Ciencias de la Universidad Autónoma del Estado de México, Instituto Literario No. 100, Toluca, C.P. 50000, Edo de Mex, México*

We use the Sáez-Ballester theory on anisotropic Bianchi I cosmological model, with barotropic fluid and cosmological constant. We obtain the classical solution by using the Hamilton-Jacobi approach. Also the quantum regime is constructed and exact solutions to the Wheeler-DeWitt equation are found.

Keywords: Classical and quantum exact solutions; cosmology.

Usamos la teoría de Sáez-Ballester en el modelo anisotrópico Bianchi I con un fluido barotrópico y constante cosmológica. Obtenemos las soluciones clásicas usando el enfoque de Hamilton-Jacobi. El regimen cuántico también es construido y soluciones exactas a la ecuación de Wheeler-DeWitt son encontradas.

Descriptores: Soluciones clásicas y cuánticas exactas; cosmología.

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I. INTRODUCTION

Saez and Ballester [1] formulated a scalar-tensor theory of gravitation in which the metric is coupled with a dimensionless scalar field. In this direction, many works in the classical regime have been done [2–5], yet a study of the anisotropy behaviour trough the form introduced in the line element has been conected. In this theory the strenght of the coupling between gravity and the scalar field is determined by an arbitrary coupling function ω. In spite the dimensionless character of the scalar field, an antigravity regime appears, this suggests a possible way to solve the missing matter problem in non-flat FRW cosmologies. However, Armendariz-Picon et al called this scenario as *K-essence* [6], which is characterized by a scalar field with a non-canonical kinetic energy. Usually K-essence models are restricted to the lagrangian density of the form

$$\mathrm{S} = \int \mathrm{d}^4\mathrm{x}\,\sqrt{-\mathrm{g}}\,\mathrm{f}(\phi)\,\left(\nabla\phi\right)^2\,, \tag{1}$$

one of the motivations to consider this type of lagrangian originates from string theory [7]. For more details for K-essence applied to dark energy, you can see [8] and reference therein.

On another front, the quantization program of this theory has not ben constructed, because should be dark to build the ADM formalism. Thus, we transform this theory to conventional one, where the dimensionless scalar field is obtained from energy-momentum tensor as a exotic matter, and in this sense, we can use this structure for the

*Electronic address: socorro@fisica.ugto.mx
†Electronic address: msabido@fisica.ugto.mx

quantization program, where the ADM formalism is well known for different classes of matter [9]

In this work, we use this formulation to obtain classical and quantum exact solutions to anisotropic Bianchi type I cosmological model, including a cosmological constant Λ. The first step is to write Sáez-Ballester formalism in the usual manner, that is, we calculate the corresponding energy-momentum tensor to the scalar field and give the equivalent lagrangian density. Next, we proceed to obtain the corresponding canonical lagrangian $\mathcal{L}_{can}$ to Bianchi type I through the lagrange transformation, we calculate the classical hamiltonian $\mathcal{H}$, from which we find the Wheeler-DeWitt (WDW) equation of the corresponding cosmological model under study. We employ in this work the Misner parametrization due that in natural way appear the anisotropy parameters to the scale factors, and we can to analyze its behaviour in easy way.

The more simple generalization to lagrangian density for the Sáez-Ballester theory [1] with cosmological term, is

$$\mathcal{L}_{\rm geo} = \left(\mathrm{R} - 2\Lambda - \mathrm{F}(\phi)\phi_{,\gamma}\phi^{,\gamma}\right), \tag{2}$$

where $\phi^{,\gamma} = g^{\gamma\alpha}\phi_{,\alpha}$, R the scalar curvature, $F(\phi)$ a dimensionless functional scalar field. In classical field theory with scalar field, this formalism corresponds to null potencial in the field ϕ, but the kinetic term is exotic by the factor $F(\phi)$.

From the lagrangian (2) we can build the complete action

$$\mathrm{I} = \int_\Sigma \sqrt{-\mathrm{g}}(\mathcal{L}_{\rm geo} + \mathcal{L}_{\rm mat})\mathrm{d}^4\mathrm{x}, \tag{3}$$

where $\mathcal{L}_{mat}$ is the matter lagrangian, g is the determinant of metric tensor. The field equations for this theory are

$$\begin{aligned} \mathrm{G}_{\alpha\beta} + \mathrm{g}_{\alpha\beta}\Lambda - \mathrm{F}(\phi)\left(\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}\mathrm{g}_{\alpha\beta}\phi_{,\gamma}\phi^{,\gamma}\right) &= 8\pi\mathrm{G}\mathrm{T}_{\alpha\beta}, \\ 2\mathrm{F}(\phi)\phi^{,\alpha}_{;\alpha} + \frac{\mathrm{dF}}{\mathrm{d}\phi}\phi_{,\gamma}\phi^{,\gamma} &= 0, \end{aligned} \tag{4}$$

where G is the gravitational constant and as usual the semicolon means a covariant derivative.

These set of equations (4) can be obtained using the equivalent lagrangian as a matter and energy-momentum tensor for this field ϕ,

$$\begin{aligned} \mathcal{L}_\phi &= \mathrm{F}(\phi)\mathrm{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}, \\ \mathrm{T}_{\alpha\beta}(\phi) &= \mathrm{F}(\phi)\left(\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}\mathrm{g}_{\alpha\beta}\phi_{,\gamma}\phi^{,\gamma}\right), \end{aligned} \tag{5}$$

in this way, we write the action (3) in the usual form

$$\mathrm{I} = \int_\Sigma \sqrt{-\mathrm{g}}\left(\mathrm{R} - 2\Lambda + \mathcal{L}_{\rm mat} + \mathcal{L}_\phi\right)\mathrm{d}^4\mathrm{x}, \tag{6}$$

and consequently, the classical equivalence between the two theories. We can infer that this correspondence also is satisfied in the quantum regimen, because only is modified the hamiltonian constraint [9].

This work is arrangede as follow. In section II we construct the hamiltonian density for the cosmological model. In section III the classical solutions using the Hamilton-Jacobi formalism are found. Here, we have used a barotropic perfect fluid as a matter content and a cosmological constant, obtaining the solutions for differents epoch of the evolution for this cosmological model. In Section IV the cuantization scheme, obtaining the corresponding Wheeler-DeWitt equation and its solutions for different values for the γ parameter. Finally, the section V is devoted to discussion.

II. THE HAMILTONIAN DENSITY

The line element for the cosmological Bianchi type I has the form

$$ds^2 = -N^2 dt^2 + e^{2\Omega+2\beta_+ +2\sqrt{3}\beta_-}\left(dx^1\right)^2 + e^{2\Omega+2\beta_+ -2\sqrt{3}\beta_-}\left(dx^2\right)^2 + e^{2\Omega-4\beta_+}\left(dx^3\right)^2 \qquad (7)$$

where N is the lapse function, $\beta_\pm$ are the corresponding anisotropic parameter in the scale factors, Ω play the role as the scale factor like to flat Friedmann-Robertson-Walker cosmological model ($e^\Omega \equiv A$). The total volume for all diagonal Bianchi cosmological models is given by the expression $V = e^{3\Omega(t)}$,that will appear in the solutions for all parameter in this theory.

Then, the corresponding lagrangian density in this theory is

$$\mathcal{L} = \frac{6\dot{\Omega}^2 e^{3\Omega}}{N} - 6\frac{\dot{\beta}_+^2 e^{3\Omega}}{N} - 6\frac{\dot{\beta}_-^2 e^{3\Omega}}{N} + \frac{F(\phi)}{N}\dot{\phi}^2 e^{3\Omega} + \left(16N\pi G\rho - 2N\Lambda\right)e^{3\Omega}.$$

wich can be rewritten in the canonical form,

$$\mathcal{L}_{can} = \Pi_\Omega \dot{\Omega} + \Pi_{\beta_+}\dot{\beta}_+ + \Pi_{\beta_-}\dot{\beta}_- + \Pi_\phi \dot{\phi} - N\mathcal{H}, \qquad (8)$$

with $\mathcal{H}$ as the hamiltonian density, and the momentas are defined in the usual way $\Pi_{q^i} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$, where $q^i = (\Omega, \beta_+, \beta_-, \phi)$ are the field coordinates for this system,

$$\Pi_\Omega = \frac{\partial \mathcal{L}}{\partial \dot{\Omega}} = 12\frac{\dot{\Omega}e^{3\Omega}}{N}, \quad \rightarrow \quad \frac{\Pi_\Omega}{2} = 6\frac{\dot{\Omega}e^{3\Omega}}{N} \quad \rightarrow \quad \dot{\Omega} = Ne^{-3\Omega}\frac{\Pi_\Omega}{12}, \qquad (9)$$

$$\Pi_{\beta_+} = \frac{\partial \mathcal{L}}{\partial \dot{\beta}_+} = -12\frac{\dot{\beta}_+ e^{3\Omega}}{N}, \quad \rightarrow \quad \frac{\Pi_{\beta_+}}{2} = -6\frac{\dot{\beta}_+ e^{3\Omega}}{N} \quad \rightarrow \quad \dot{\beta}_+ = -Ne^{-3\Omega}\frac{\Pi_{\beta_+}}{12}, \qquad (10)$$

$$\Pi_{\beta_-} = \frac{\partial \mathcal{L}}{\partial \dot{\beta}_-} = -12\frac{\dot{\beta}_- e^{3\Omega}}{N}, \quad \rightarrow \quad \frac{\Pi_{\beta_-}}{2} = -6\frac{\dot{\beta}_+ e^{3\Omega}}{N} \quad \rightarrow \quad \dot{\beta}_- = -Ne^{-3\Omega}\frac{\Pi_{\beta_-}}{12}, \qquad (11)$$

$$\Pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 2\frac{F(\phi)\dot{\phi}e^{3\Omega}}{N}, \quad \rightarrow \quad \frac{\Pi_\phi}{2} = \frac{F(\phi)\dot{\phi}e^{3\Omega}}{N} \quad \rightarrow \quad \dot{\phi} = \frac{N\Pi_\phi}{2F(\phi)e^{3\Omega}}. \qquad (12)$$

The matter is introduced as a barotropic perfect fluid $P = \gamma\rho$ with γ a constant between $-1 < \gamma < 1$, who energy-momentum tensor is given by the following expression $T_{\mu\nu} = (\rho + P)U_\mu U_\nu - g_{\mu\nu}P$ where U_μ is the four-velocity, ρ is the energy density and P is the thermodynamic pressure in the fluid, respectively. Using the covariant estructure in this tensor, we obtain the differential equation $3\dot{\Omega}\rho + 3\dot{\Omega}P + \dot{\rho} = 0$ and the solution

$$\rho = \mu_\gamma e^{-3\Omega(1+\gamma)}.$$

with μ_γ a constant for the corresponding scenario.

So, we obtain the following expresion for the canonical lagrangian

$$\mathcal{L}_{can} = \Pi_{q^i}\dot{q}^i - \frac{Ne^{-3\Omega}}{24}\left[\Pi_\Omega^2 - \Pi_{\beta_+}^2 - \Pi_{\beta_-}^2 + \frac{6\Pi_\phi^2}{F(\phi)} - 384\pi G\mu_\gamma e^{3\Omega(1-\gamma)} + 48\Lambda e^{6\Omega}\right],$$

and

$$\mathcal{H} = \frac{e^{-3\Omega}}{24}\left(\Pi_\Omega^2 - \Pi_{\beta_+}^2 - \Pi_{\beta_-}^2 + \frac{6\Pi_\phi^2}{F(\phi)} - 384\pi G\mu_\gamma e^{3\Omega(1-\gamma)} + 48\Lambda e^{6\Omega}\right), \qquad (13)$$

and using the lagrange equation for the field N, we get the Hamiltonian constraint

$$\mathcal{H} = \Pi_\Omega^2 - \Pi_{\beta_+}^2 - \Pi_{\beta_-}^2 + \frac{6\Pi_\phi^2}{F(\phi)} - 384\pi G\mu_\gamma e^{3\Omega(1-\gamma)} + 48\Lambda e^{6\Omega} = 0. \qquad (14)$$

III. CLASSICAL REGIMEN: HAMILTON-JACOBI APPROACH

Employing the Hamilton-Jacobi formulation, where the momentas are $\Pi_q = \frac{\partial S_q}{\partial q}$, where S_q is the superpotential function, the hamiltonian takes the following form

$$\left(\frac{\partial S_\Omega}{\partial \Omega}\right)^2 - \left(\frac{\partial S_{\beta_+}}{\partial \beta_+}\right)^2 - \left(\frac{\partial S_{\beta_-}}{\partial \beta_-}\right)^2 + \frac{6}{F(\phi)}\left(\frac{\partial S_\phi}{\partial \phi}\right)^2 - 384\pi G \mu_\gamma e^{3\Omega(1-\gamma)} + 48\Lambda e^{6\Omega} = 0, \tag{15}$$

solving for the variable Ω we have

$$\left(\frac{\partial S_\Omega}{\partial \Omega}\right)^2 - 384\pi G \mu_\gamma e^{3\Omega(1-\gamma)} + 48\Lambda e^{6\Omega} = \left(\frac{\partial S_{\beta_+}}{\partial \beta_+}\right)^2 + \left(\frac{\partial S_{\beta_-}}{\partial \beta_-}\right)^2 - \frac{6}{F(\phi)}\left(\frac{\partial S_\phi}{\partial \phi}\right)^2 = \xi^2$$

using (9), we obtain the following integral equation depending on time parameter $N dt = d\tau$, we have

$$\frac{dS_\Omega}{d\Omega} = \sqrt{\xi^2 + 384\pi G \mu_\gamma e^{3\Omega(1-\gamma)} - 48\Lambda e^{6\Omega}} = 12 e^{3\Omega}\frac{d\Omega}{d\tau} \tag{16}$$

$$\Delta\tau = \int \frac{12}{\sqrt{\xi^2 e^{-6\Omega} + 384\pi G \mu_\gamma e^{-3\Omega(1+\gamma)} - 48\Lambda}} d\Omega \tag{17}$$

where ξ is a separation constant. This equation do not have a general solution, but we can solve for the particular values of the γ parameter, which we present in the Table I.

Case	$\Omega(\tau)$
Inflation, $\gamma = -1$ $b_{-1} = 384\pi G\mu_{-1} - 48\Lambda > 0$	$\frac{1}{3}\mathrm{Ln}\left[\frac{1}{4\sqrt{b_{-1}}}\left(e^{\frac{\sqrt{b_{-1}}}{4}\Delta\tau} - 4\xi^2 e^{-\frac{\sqrt{b_{-1}}}{4}\Delta\tau}\right)\right]$
Dust, $\gamma = 0$ $b_0 = 384\pi G\mu_0, \quad \Lambda < 0$	$\frac{1}{3}\mathrm{Ln}\left[\frac{\left(e^{\sqrt{-3\Lambda}\Delta\tau} - \frac{b_0}{4\sqrt{-3\Lambda}}\right)^2 - 4\xi^2}{16\sqrt{-3\Lambda}e^{\sqrt{-3\Lambda}\Delta\tau}}\right]$
Stiff matter, $\gamma = 1$ $b_1 = \xi^2 + 384\pi G\mu_1$, $\Lambda < 0$	$\frac{1}{3}\mathrm{Ln}\left[\frac{1}{16\sqrt{-3\Lambda}}\left(e^{\sqrt{-3\Lambda}\Delta\tau} - 4b_1 e^{-\sqrt{-3\Lambda}\Delta\tau}\right)\right]$

Table I. Solutions for Ω in the scenarios $\gamma = -1, 0, 1$

The corresponding solutions for the anisotropic functions and the field ϕ appear in the Table II. For the field ϕ we consider

$$\frac{6}{F(\phi)}\left(\frac{\partial S_\phi}{\partial \phi}\right)^2 = -\xi^2 + \kappa_+^2 + \kappa_-^2 = \theta^2$$

$$\frac{dS_\phi}{d\phi} = \sqrt{\frac{F(\phi)}{6}}\theta \equiv -2F(\phi)\frac{d\phi}{d\tau}e^{3\Omega}$$

with $\theta^2 = -\xi^2 + \kappa_+^2 + \kappa_-^2$, with the cuadrature solutions

$$\int \sqrt{F(\phi)}\, d\phi = \frac{\theta}{2\sqrt{6}}\int e^{-3\Omega}(\tau)\, d\tau, \tag{18}$$

Now considering the original Sáez-Ballester theory, $F(\phi) = \omega\phi^m$, the corresponding solution for all m are

Case	$\beta_\pm(\tau)$	$\phi(\tau)$
$\gamma=-1,\ \omega>0$ $k_+=-\sqrt{3}k_-$	$\mp\frac{2\kappa_\pm}{3\xi}\tanh^{-1}\left(\frac{\mathrm{e}^{\frac{\sqrt{\mathrm{b}_{-1}}}{4}\Delta\tau}}{2\xi}\right)$	$\left[\mp\frac{2\theta(\mathrm{m}+2)}{\xi\sqrt{6\omega}}\tanh^{-1}\left(\frac{\mathrm{Exp}\left(\frac{\sqrt{\mathrm{b}_{-1}}}{4}\Delta\tau\right)}{2\xi}\right)\right]^{\frac{2}{\mathrm{m}+2}}$, $m\neq-2$ $\mathrm{Exp}\left[\mp\frac{4\theta}{\xi\sqrt{6\omega}}\tanh^{-1}\left(\frac{\mathrm{Exp}\left(\frac{\sqrt{\mathrm{b}_{-1}}}{4}\Delta\tau\right)}{2\xi}\right)\right]$, m=-2
$\gamma=0$ $\omega>0$	$\pm\frac{2\kappa_\pm}{3\xi}\tanh^{-1}\left(\frac{-\mathrm{b}_0+4\sqrt{-3\Lambda}\mathrm{e}^{\sqrt{-3\Lambda}\Delta\tau}}{8\sqrt{-3\Lambda}\xi}\right)$	$\left[\pm\frac{2\theta(\mathrm{m}+2)}{\xi\sqrt{6\omega}}\tanh^{-1}\left(\frac{-\mathrm{b}_0+4\sqrt{-3\Lambda}\mathrm{e}^{\sqrt{-3\Lambda}\Delta\tau}}{8\sqrt{-3\Lambda}\xi}\right)\right]^{\frac{2}{\mathrm{m}+2}}$, $m\neq-2$ $\mathrm{Exp}\left[\pm\frac{4\theta}{\xi\sqrt{6\omega}}\tanh^{-1}\left(\frac{-\mathrm{b}_0+4\sqrt{-3\Lambda}\mathrm{e}^{\sqrt{-3\Lambda}\Delta\tau}}{8\sqrt{-3\Lambda}\xi}\right)\right]$, m=-2,
$\gamma=1,\ \omega>0$ $k_+=-\sqrt{3}k_-$	$\mp\frac{2\kappa_\pm}{3\sqrt{\mathrm{b}_1}}\tanh^{-1}\left(\frac{\mathrm{e}^{\sqrt{-3\Lambda}\tau}}{2\sqrt{\mathrm{b}_1}}\right)$	$\left[\mp\frac{2\theta(\mathrm{m}+2)}{\sqrt{6\mathrm{b}_1\omega}}\tanh^{-1}\left(\frac{\mathrm{e}^{\sqrt{-3\Lambda}\Delta\tau}}{2\sqrt{\mathrm{b}_1}}\right)\right]^{\frac{2}{\mathrm{m}+2}}$, $m\neq-2$ $\mathrm{Exp}\left[\mp\frac{4\theta}{\sqrt{6\mathrm{b}_1\omega}}\tanh^{-1}\left(\frac{\mathrm{e}^{\sqrt{-3\Lambda}\Delta\tau}}{2\sqrt{\mathrm{b}_1}}\right)\right]$, m=-2

Table II. Solutions for the anisotropic variables $\beta_\pm$ and field ϕ in the scenarios $\gamma=-1,0,1$

These set of solutions satisfy the Einstein field equation (4), which were checked using REDUCE package. It is interesting to note the behaviour to the cosmological constant in this theory. In the inflationary scenario, we have a positive value in such a way that the universe has a bigger growth (but next change to negative one, where the universe have a small growth).

IV. QUANTUM REGIMEN: WHEELER-DEWITT EQUATION

For quantum regime, we calculate the Wheeler-DeWitt equation

$$\hat{\mathrm{H}}\Psi=\left\{\hat{\Pi}_\Omega^2-\hat{\Pi}_{\beta_+}^2-\hat{\Pi}_{\beta_-}^2+\frac{6\hat{\Pi}_\phi^2\phi^{-\mathrm{m}}}{\omega}-384\pi\mathrm{G}\mu_\gamma\mathrm{e}^{3\Omega(1-\gamma)}+48\Lambda\mathrm{e}^{6\Omega}\right\}\Psi=0$$

where the momenta operators $\hat{\Pi}_q=-i\hbar\frac{\partial}{\partial q}$, Ψ is the wave function of the universe, also we choose $\hbar=1$, thus

$$\hat{H}\Psi=\left\{\left(-\frac{\partial^2}{\partial\Omega^2}+Q\frac{\partial}{\partial\Omega}\right)+\frac{\partial^2}{\partial\beta_+^2}+\frac{\partial^2}{\partial\beta_-^2}-\frac{6\phi^{-m}}{\omega}\frac{\partial^2}{\partial\phi^2}-384\pi G\mu_\gamma e^{3\Omega(1-\gamma)}+48\Lambda e^{6\Omega}\right\}\Psi=0,$$

where we have used $\left(-\frac{\partial^2}{\partial\Omega^2}+Q\frac{\partial}{\partial\Omega}\right)$ for solving the factor ordering problem. Applying the separation method, using for the wave function

$$\Psi=\mathcal{A}(\Omega)\,\mathcal{B}(\beta_+)\,\mathcal{C}(\beta_-)\,\mathcal{D}(\phi)$$

we obtain

$$\left\{-\frac{1}{\mathcal{A}}\frac{d^2\mathcal{A}}{d\Omega^2}+Q\frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{d\Omega}+\frac{1}{\mathcal{B}}\frac{d^2\mathcal{B}}{d\beta_+^2}+\frac{1}{\mathcal{C}}\frac{d^2\mathcal{C}}{d\beta_-^2}-\frac{1}{\mathcal{D}}\frac{6\phi^{-m}}{\omega}\frac{d^2\mathcal{D}}{d\phi^2}-384\pi G\mu_\gamma e^{3\Omega(1-\gamma)}+48\Lambda e^{6\Omega}\right\}=0,$$

yielding the following set of differential equations

$$
\begin{aligned}
-\frac{1}{\mathcal{A}}\frac{\mathrm{d}^2\mathcal{A}}{\mathrm{d}\Omega^2}+\mathrm{Q}\frac{1}{\mathcal{A}}\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\Omega}-384\pi\mathrm{G}\mu_\gamma\mathrm{e}^{3\Omega(1-\gamma)}+48\Lambda\mathrm{e}^{6\Omega} &= -\mathrm{a}_1^2, &(19)\\
\frac{1}{\mathcal{B}}\frac{\mathrm{d}^2\mathcal{B}}{\mathrm{d}\beta_+^2} &= \mathrm{a}_2^2, &(20)\\
\frac{1}{\mathcal{C}}\frac{\mathrm{d}^2\mathcal{C}}{\mathrm{d}\beta_-^2} &= \mathrm{a}_3^2, &(21)\\
\frac{1}{\mathcal{D}}\frac{6\phi^{-\mathrm{m}}}{\omega}\frac{\mathrm{d}^2\mathcal{D}}{\mathrm{d}\phi^2} &= \mathrm{a}_4^2, &(22)
\end{aligned}
$$

where $\mathrm{a}_4^2=-\mathrm{a}_1^2+\mathrm{a}_2^2+\mathrm{a}_3^2$, with a_i^2 separation constants. The choose of sign for these constants is arbitrary, in absence to initial conditions of our universe, studied under this cosmological model.

The solution for the equations (20, 21) have the generic form

$$
\mathcal{B}=\mathrm{e}^{\pm \mathrm{a}_2\beta_+}, \qquad \mathcal{C}=\mathrm{e}^{\pm \mathrm{a}_3\beta_-}, \tag{23}
$$

The solution for the equation (22) is more complicated, because depend to the constant m, which is a parameter to the Sáez-Ballester theory. This equation is rewritten as $\frac{\mathrm{d}^2\mathcal{D}}{\mathrm{d}\phi^2}-\frac{\omega \mathrm{a}_4^2\phi^{\mathrm{m}}}{6}\mathcal{D}=0$, which is analog to the equation find in the reference [10], $y''-ax^ny=0$, with $a=\frac{\omega a_4^2}{6}$ and $n=m$.

1. case m=-2 correspond to the Euler equation, who solution have the following structure [10]

$$
\mathcal{D}=\sqrt{\phi}\begin{cases} \mathrm{c}_1\,\phi^{\mu}+\mathrm{c}_2\phi^{-\mu} & \mathrm{si} \quad a>-\frac{1}{4}\\ \mathrm{c}_1\;+\mathrm{c}_2\mathrm{Ln}\phi & \mathrm{si} \quad a=-\frac{1}{4}\\ \mathrm{c}_1\sin(\mu\mathrm{Ln}\phi)+\mathrm{c}_2\cos(\mu\mathrm{Ln}\phi) & \mathrm{si} \quad a<-\frac{1}{4}\end{cases} \tag{24}
$$

   where $\mu=\frac{1}{2}\sqrt{|1+4a|}>0$

2. In the case m=-4, we introduce the following transformation $z=\frac{1}{\phi}$ and $\frac{\mathrm{u}}{\mathrm{z}}=\mathcal{D}$, yielding to differential equation more easy to solve $\frac{\mathrm{d}^2\mathrm{u}}{\mathrm{dz}^2}-\mathrm{au}=0$; so the solution become as

$$
\mathcal{D}=\phi\begin{cases} \mathrm{c}_1\sinh(\sqrt{\mathrm{a}}\phi)+\mathrm{c}_2\cosh(\sqrt{\mathrm{a}}\phi) & \mathrm{si} \quad a>0\\ \mathrm{c}_1\sin\left(\sqrt{|\mathrm{a}|}\phi\right)+\mathrm{c}_2\cos\left(\sqrt{|\mathrm{a}|}\phi\right) & \mathrm{si} \quad a<0\end{cases} \tag{25}
$$

   the case a=0, is descarted, because this imply that $\omega=0$, yielding to the Einstein theory.

3. When the m parameter satisfy the relation $\frac{2}{m+2}=2n+1$, where n is an integer, the general solution take the form

$$
\mathcal{D}=\phi\begin{cases} \left(\phi^{1-2\mathrm{q}}\frac{\mathrm{d}}{\mathrm{d}\phi}\right)^{\mathrm{n}+1}\left[\mathrm{D}_6\mathrm{Exp}\left(\sqrt{\frac{\omega}{6}}\frac{\phi^{\mathrm{q}}}{\mathrm{q}}\right)+\mathrm{D}_7\mathrm{Exp}\left(-\sqrt{\frac{\omega}{6}}\frac{\phi^{\mathrm{q}}}{\mathrm{q}}\right)\right] & \mathrm{si} \quad \mathrm{n}\geq 0\\ \left(\phi^{1-2\mathrm{q}}\frac{\mathrm{d}}{\mathrm{d}\phi}\right)^{-\mathrm{n}}\left[\mathrm{D}_6\mathrm{Exp}\left(\sqrt{\frac{\omega}{6}}\frac{\phi^{\mathrm{q}}}{\mathrm{q}}\right)+\mathrm{D}_7\mathrm{Exp}\left(-\sqrt{\frac{\omega}{6}}\frac{\phi^{\mathrm{q}}}{\mathrm{q}}\right)\right] & \mathrm{si} \quad \mathrm{n}<0\end{cases} \tag{26}
$$

   where D_6, D_7 are integration constants and $q=\frac{m+2}{2}=\frac{1}{2n+1}$.

4. General solution for any m, the solution is expressed in terms of the Bessel function and modified Bessel function, for the field ϕ

$$
\mathcal{D}=\sqrt{\phi}\,\mathrm{Z}_\nu\left(\frac{\sqrt{\mathrm{a}}}{\mathrm{q}}\phi^{\mathrm{q}}\right), \tag{27}
$$

where Z_ν is a generic Bessel function, $\nu = \frac{1}{2q}$ is the order to the corresponding Bessel function, $\mathrm{q} = \frac{1}{2}(\mathrm{m}+2)$. If $a < 0$ imply that $\omega < 0$, Z_ν become the modified Bessel function, (I_ν, K_ν). When $a > 0, \rightarrow w > 0$, $\mathrm{Z}_\nu \rightarrow (\mathrm{J}_\nu, \mathrm{Y}_\nu)$.

On the other hand, the equation (19) does have not general solution, then this is solved for particular cases in the γ parameter, and for this, is rewritten in the following form

$$\frac{\mathrm{d}^2\mathcal{A}}{\mathrm{d}\Omega^2} - \mathrm{Q}\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\Omega} + \left(384\pi \mathrm{G}\mu_\gamma \mathrm{e}^{3\Omega(1-\gamma)} - 48\Lambda \mathrm{e}^{6\Omega} - \mathrm{a}_1^2\right)\mathcal{A} = 0, \tag{28}$$

1. Any factor ordering Q and the inflation phenomenom $\gamma = -1$

$$\frac{\mathrm{d}^2\mathcal{A}}{\mathrm{d}\Omega^2} - \mathrm{Q}\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\Omega} + \left[\mathrm{b}_{-1}\mathrm{e}^{6\Omega} - \mathrm{a}_1^2\right]\mathcal{A} = 0, \qquad \mathrm{b}_{-1} = 384\pi\mathrm{G}\mu_{-1} - 48\Lambda \tag{29}$$

making the transformations $z = \frac{\sqrt{b_1}}{3}e^{3\Omega}$ and $\mathcal{A} = z^{\frac{Q}{6}}\Phi(z)$ we arrive at Bessel differential equation for the function Φ. With this the general solution become [10]

$$\mathcal{A} = \left(\frac{\sqrt{\mathrm{b}_{-1}}}{3}\mathrm{e}^{3\Omega}\right)^{\frac{Q}{6}} \mathrm{Z}_\nu\left(\frac{\sqrt{\mathrm{b}_{-1}}}{3}\mathrm{e}^{3\Omega}\right), \qquad \nu = \pm\frac{1}{6}\sqrt{\mathrm{Q}^2 + 4\mathrm{a}_1^2} \tag{30}$$

where Z_ν is a generic Bessel function. If $\mathrm{b}_{-1} > 0$, we have the ordinary Bessel function, in other case, will be the modified Bessel function.

2. factor ordering Q=0 and $\gamma = 0$

$$\frac{\mathrm{d}^2\mathcal{A}}{\mathrm{d}\Omega^2} - \left(48\Lambda\mathrm{e}^{6\Omega} - \mathrm{b}_0\mathrm{e}^{3\Omega} + \mathrm{a}_1^2\right)\mathcal{A} = 0, \tag{31}$$

making the transformation $\mathrm{z} = \mathrm{e}^{3\Omega}$ and $R = z^{-\frac{a_1}{3}}\mathcal{A}$, is carried to equation $9\mathrm{z}\frac{\mathrm{d}^2\mathrm{R}}{\mathrm{dz}^2} + 9\left(\frac{2}{3}\mathrm{a}_1 + 1\right)\frac{\mathrm{dR}}{\mathrm{dz}} - (48\Lambda\mathrm{z} - \mathrm{b}_0)\,\mathrm{R} = 0$ who solution is constructed by the degenerate hypergeometric function $\mathrm{F}_1(\mathrm{a}, \mathrm{b}; \mathrm{z})$ [10]

$$\mathcal{A} = \mathrm{e}^{\mathrm{a}_1\Omega}\mathrm{Exp}\left[\frac{2}{3}\sqrt{6\Lambda}\mathrm{e}^{\mathrm{a}_1\Omega}\right]\,\mathrm{F}_1\left(\frac{\mathrm{B(k)}}{18\mathrm{k}}, \frac{2}{3}\mathrm{a}_1 + 1; \frac{\mathrm{e}^{3\Omega}}{\lambda}\right), \tag{32}$$

where $\lambda = -\frac{1}{2k}$, $\mathrm{k} = \frac{2}{3}\sqrt{6\Lambda}$, $\mathrm{B(k)} = 9\mathrm{k}(\frac{2}{3}\mathrm{a}_1 + 1) + \mathrm{b}_0$.

3. Any factor ordering and stiff matter $\gamma = 1$

$$\frac{\mathrm{d}^2\mathcal{A}}{\mathrm{d}\Omega^2} - \mathrm{Q}\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\Omega} + \left[\mathrm{b}_1 - 48\Lambda\mathrm{e}^{6\Omega}\right]\mathcal{A} = 0, \qquad \mathrm{b}_1 = 384\pi\mathrm{G}\mu_1 - \mathrm{a}_1^2 \tag{33}$$

in similar way that the first case, the transformations $z = 4\sqrt{-\frac{\Lambda}{3}}\,e^{3\Omega}$ and $\mathcal{A} = z^{\frac{Q}{6}}\Phi(z)$, the differential Bessel function appear for the function Φ, then we have the general solution

$$\mathcal{A} = \left(4\sqrt{-\frac{\Lambda}{3}}\,e^{3\Omega}\right)^{\frac{Q}{6}} Z_\nu\left(4\sqrt{-\frac{\Lambda}{3}}\,e^{3\Omega}\right), \qquad \nu = \pm\frac{1}{6}\sqrt{Q^2 - 4b_1}. \tag{34}$$

with $\Lambda < 0$, having the ordinary Bessel function; In the case when the factor ordering become Q=0, we have the same Bessel function, but the cosmological constant could be positive or negative, yielding to modified Bessel function and ordinary Bessel function, respectively, and the order become imaginary in both cases $\nu = \pm i\frac{\sqrt{b_1}}{3}$.

V. CONCLUSIONS

One equivalent density lagrangian was build in order to apply the quantum regime in the Sáez-Ballester theory, in where the constant ω can be used to have a lorenzian (-1,1,1,1) or seudo-lorenzian (-1,-1,1,1) signature when we build the Wheeler-DeWitt equation. The values for this parameter in the classical one is dictated when we apply the condition that we must have real functions, which is encoded in the parameter a, equations (24,25). In this sense, the classical and quantum exact solutions were found for the cosmological Bianchi type I model in the frame of Sáez-Ballester theory for the scenarius in the γ parameter $\{-1, 0, 1\}$. The presense of the exotic field ϕ does not delay the anisotropic behaviour in this model. Moreless, the classical behaviour of this field for large value in the parameter ω, it is similar to the anisotropic parameters $\beta_\pm$.

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